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Geometric form of volcanoes with a limited based

A. Arjona[1], J.I. Díaz[2], J. Fernández[1],

[1] *Instituto de Astronomía y Geodesia, CSIC-UCM, Plaza de Ciencias, 3. Madrid. E-mails:* alicia_arjona@mat.ucm.es, jose_fernandez@mat.ucm.es.
[2] *Dpto. de Matemática Aplicada, Univ. Complutense de Madrid. Plaza de Ciencias, 3. Madrid. E-mail:*ji_diaz@mat.ucm.es.

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Abstract

Many volcanic constructs have geometric different shapes depending on different phenomena as parasitic cones, erosion or coral growth. In Lacey, Ockendon and Turcotte [11] the authors proposed a nonlinear model proving that the shape of volcanoes is determined by the hydraulic resistance to the flow of magma, from a line source, through the porous edifice. This model was later extended in Angevine, Turcotte and Ockendon [2] to include the shape of aseismic, submarine ridges. In this communication we propose a modification of the above mentioned models in order to simulate the more realistic case of volcanoes with a limited base.

We start by proving that the free boundary (the volcano base) associated to the models described in the above mentioned references is not bounded as $t \to +\infty$ (even if it is assumed that the flux generated by the magma supply $Q_0(t)$ in the line source is a bounded function). As said before, this unrealistic fact (specially in the case of volcanoes located in islands) is the main reason to propose a modification of the involved nonlinear equations in order to obtain a new model giving rise to a bounded free boundary (even if $t \to +\infty$). By using some suitable variations of the modelling arguments of Angevine, Turcotte and Ockendon [2] and Lacey, Ockendon and Turcotte [11] we propose the new model,

$$\begin{cases} \frac{\partial H}{\partial t} = K\frac{\partial^2 H^2}{\partial x^2} + \frac{\mu x}{|x|}\frac{\partial H^\lambda}{\partial x}, & x \in \mathbb{R} - \{0\}, t > 0 \\ -K\frac{\partial H^2}{\partial x}(0,t) = Q_0(t), & t > 0, \\ H(0,x) = H_0(x), & x \in \mathbb{R} - \{0\}. \end{cases} \tag{1}$$

Here we assume known the constants K, $\mu, \lambda > 0$ (which depend on the constitutive porous material) and that $Q_0(t) \geq 0$, $H_0(x) \geq 0$ and H_0 has compact support in $\mathbb{R} - \{0\}$. The models proposed in Angevine, Turcotte and Ockendon [2] and Lacey, Ockendon and Turcotte [11] correspond to the case of $\mu = 0$. We prove that when $\lambda \in (0, 2)$ and $Q_0(t)$ is a bounded function (as it corresponds to the more important examples) then, if we denote by $\xi_{\pm}(t)$ the free boundary (formed by two curves) given by support of $H(t; \cdot)$, i.e. supp $H(t; \cdot) = [\xi_-(t), 0] \cup [0, \xi_+(t)]$, necessarily $|\xi_{\pm}(t)| < \xi_\infty$ for any $t > 0$, for some $\xi_\infty < +\infty$. This conclusion leads to a better comparison between the bathymetric and theoretical profiles of many volcanoes.

1 Introduction

Let the governing equations for two-dimensional flow of uniform incompressible fluid through a rigid, isotropic porous medium were used in [2] to derive the geometrical form of aseismic volcanoes. They started from the basic equations

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial w}{\partial z} = 0, \\ u = -\dfrac{k}{\mu\phi}\dfrac{\partial p}{\partial x}, \\ w = -\dfrac{k}{\mu\phi}\left(\dfrac{\partial p}{\partial z} + \rho_m g\right), \end{cases} \tag{2}$$

where u and w are the velocities in the x and z directions of the flow, k is permeability, μ is dynamic viscosity, ϕ is porosity, p is pressure, ρ_m is magma density, and g is the gravitational acceleration. These equations are combined to get

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial z^2} = 0. \tag{3}$$

The boundary conditions considered in [2] let the following:

$$\begin{cases} z = h, & p = \rho_m g(d - h), & \text{pressure due to the overlying seawater,} \\ z = h, & w = \dfrac{\partial h}{\partial t} + u\dfrac{\partial h}{\partial x}, & \text{kinematic constraint on the upper surface,} \\ z = 0, & \dfrac{\partial p}{\partial z} = -\rho_m g, & \text{on the base of the ridges,} \end{cases} \tag{4}$$

where d is the depth of the ocean floor and $z = h(x, t)$. We also recall that the magma supply requires the additional condition

$$uh \to \frac{Q_0}{2\phi} \text{ as } x \to 0. \tag{5}$$

By introducing the small aspect ratio, a rescale is introduced originating the new terms; $Z = z/\epsilon, H = h/\epsilon, D = d/\epsilon$ and $T = t\epsilon$, with $\epsilon << 1$. Equations (3) and (4) become:

$$\begin{cases} & \epsilon^2\dfrac{\partial^2 p}{\partial x^2} + \dfrac{\partial^2 p}{\partial Z^2} = 0, \\ Z = H, & p = \epsilon\rho_m g(D - H), \\ Z = H, & w = \epsilon^2\dfrac{\partial H}{\partial T} + \epsilon u\dfrac{\partial H}{\partial x}, \\ Z = 0, & \dfrac{\partial p}{\partial Z} = -\epsilon\rho_m g, \end{cases} \tag{6}$$

Finally, after rescaling (2) and by using an expansion in the form

$$p = \epsilon p_0 + \epsilon^3 p_1 + ...$$

it was proved in [2] that the velocities at $Z = H$ must be given by

$$\begin{cases} u = -\epsilon \dfrac{k(\rho_m - \rho_w)g}{\mu\phi} \dfrac{\partial H}{\partial x}, \\ w = \epsilon^2 \dfrac{k(\rho_m - \rho_w)g}{\mu\phi} H \dfrac{\partial^2 H}{\partial x^2}. \end{cases} \tag{7}$$

Substituting these velocities into third equation of (6) they arrive to the degenerate quasi-linear equation of Boussinesq type

$$\frac{\partial H}{\partial T} = \frac{k(\rho_m - \rho_w)g}{\mu\phi} \left(H \frac{\partial H}{\partial x} \right)_x. \tag{8}$$

The solution of this equation, satisfying the associated boundary conditions to (4) and (5) were studied in [2] by using their self-similar structure. Here we shall see (Theorem 1) that if we call as $\xi(t)$ to the free boundary given by support of $H(t,\cdot) = [-\xi(t), 0) \cup (0, \xi(t)]$, for any $t > 0$, then necessarily $\xi(t) \longrightarrow +\infty$ as $t \longrightarrow +\infty$, which does not seems to be very realistic. So, if we assume symmetry conditions on the initial data $H_0(x)$, to improve the model, avowing such conclusion, we consider the system

$$P(\mu, Q_0) \equiv \begin{cases} \dfrac{\partial H}{\partial T} = K \dfrac{\partial^2 H^2}{\partial x^2} + \mu \dfrac{\partial H^\lambda}{\partial x} & x \in (0, +\infty), t > 0, \\ -KH \dfrac{\partial H}{\partial x}(0, t) = Q_0(t) & t > 0, \\ H(0, x) = H_0(x) & x \in (0, +\infty). \end{cases}$$

Notice that $P(0, Q_0)$ corresponds to the Boussinesq type equation (8). Here we assume a renormalization of the constants $K, \mu > 0$ and that $H_0(x) \geq 0$ has a compact support. We point out that a more general framework is possible (we can detail it in a subsequent draft of the paper which would contain as well the exact definition of weak solution, and other details). The main result for the new model is given in Theorem 2 and shows that if

$$0 < \lambda < 2,$$

and

$$0 \leq Q_0(t) \leq Q_{0,\infty} \text{ for any } t > 0.$$

then the support of $H(t,\cdot) = [-\xi(t), 0) \cup (0, \xi(t)]$, for any $t > 0$, but it has a limited penetration in the sense that

$$|\xi(t)| \leq \xi_\infty \text{ for any } t \geq 0,$$

for some finite $\xi_\infty < \infty$ depending on $\lambda, K, \mu, Q_{0,\infty}$ and $H_0(x)$. We mention that without the symmetry assumption on $H_0(x)$ we must work on the spatial domain $\mathbb{R} - \{0\}$ and by replacing the nonlinear pde by

$$\frac{\partial H}{\partial T} = K \frac{\partial^2 H^2}{\partial x^2} + \frac{\mu x}{|x|} \frac{\partial H^\lambda}{\partial x}.$$

The new modelling argument consists in introducing the new velocities:

$$
\begin{aligned}
u &= \epsilon \frac{k(\rho_m - \rho_w)g}{\mu\phi}\frac{\partial H}{\partial x} - \nu(\epsilon)(\frac{\rho_m - \rho_w}{\mu\phi})g\lambda H^{\lambda-1}, \\
w &= \epsilon^2 \frac{k(\rho_m - \rho_w)g}{\mu\phi} H \frac{\partial^2 H}{\partial x^2},
\end{aligned}
\tag{9}
$$

for some $0 < \lambda < 2$ (the justification of this new exponent λ may come from some other terms in the asymptotic expansion or from other form of the boundary conditions). Notice also that if $\lambda \in (1, 2)$ the new term is small for $H \in (0, H_0)$, if $\lambda \in (0, 1)$ the new term is very big if $H \in (0, H_0)$ and, which is more useful, when $\lambda = 1$ the new term is a constant. We also point out that

$$
\begin{aligned}
\epsilon^2 \frac{\partial H}{\partial t} = w - \epsilon u \frac{\partial H}{\partial x} &= \epsilon^2 \frac{k(\rho_m - \rho_w)g}{\mu\phi} H \frac{\partial^2 H}{\partial x^2} + \epsilon^2 \frac{k(\rho_m - \rho_w)g}{\mu\phi}\left(\frac{\partial H}{\partial x}\right)^2 \\
&\quad + \nu(\epsilon)\epsilon(\frac{\rho_m - \rho_w}{\mu\phi})g \frac{H^{\lambda-1}}{\lambda}\frac{\partial H}{\partial x}.
\end{aligned}
\tag{10}
$$

So,

$$
\epsilon^2 \frac{\partial H}{\partial t} = \epsilon^2 \frac{k(\rho_m - \rho_w)g}{\mu\phi}\frac{\partial}{\partial x}\left(H\frac{\partial H}{\partial x}\right) + \nu(\epsilon)\epsilon\left(\frac{\rho_m - \rho_w}{\mu\phi}\right) g \frac{\partial}{\partial x}(H^\lambda), \tag{11}
$$

and then, we must assume that $\nu(\epsilon) = \epsilon$.

2 Unlimited volcanoes base according the previous model ($\mu = 0$).

We are going to prove that the free boundary is not bounded as $t \longrightarrow +\infty$, for this we are going to prove next theorem.

Theorem 1 *Let $\zeta(t)$ the free boundary of the problem $P(0, Q_0)$, then $\zeta(t) \longrightarrow +\infty$ if $t \longrightarrow +\infty$.*

We shall built the proof in two different steps. In a first one, we shall prove that if $U(t,x)$ and $H(t,x)$ are solutions of the respective problems $P(0, Q_0)$ and $P(0, 0)$ with the same initial data then, we have $U \leq H$.

In a second step, we shall prove that if $\zeta(t)$ and $\xi(t)$ are the free boundaries of the problems $P(0, Q_0)$ and $P(0, 0)$, with the same initial data, then $\zeta(t) \longrightarrow +\infty$ as $t \longrightarrow +\infty$. This will conclude the proof since, the first step proves that $0 < \zeta(t) \leq \xi(t)$, and thus, necessarily, $\xi(t) \longrightarrow +\infty$ if $t \longrightarrow +\infty$.

The first step is a special conclusion of a more general statement:

Proposition 1 *Let H_1 and H_2 be the solutions of $P(\mu, Q_0)$ corresponding to $\mu \geq 0$, $Q_{1,0}$, $Q_{2,0}$ and $H_{1,0}$, $H_{2,0}$ respectively. Then, for any $t > 0$ we have*

$$
\int_\Omega (H_1(t,x) - H_2(t,x))_+ dx \leq \int_\Omega (H_{1,0}(x) - H_{2,0}(x))_+ dx + \int_0^t (Q_{1,0}(\tau) - Q_{2,0}(\tau))_+ d\tau \tag{12}
$$

where, we used the notation, $a_+(x) = max(0, a(x))$, for any general function defined on Ω.

Notice that as a direct consequence of the Proposition 1, and of the fact that $a_+(x) = 0$ implies that $a(x) \leq 0$, we have

Corollary 1 *Let H_1 and H_2 be the solutions of $P(\mu, Q_0)$ corresponding to $\mu \geq 0$, $Q_{1,0}$, $Q_{2,0}$ and $H_{1,0}$, $H_{2,0}$ respectively such that $Q_{1,0}(t) \leq Q_{2,0}(t)$ for any $t > 0$ and $H_{1,0}(x) \leq H_{2,0}(x)$ for $x \in \Omega$. Then $H_1(t, x) \leq H_2(t, x)$ for any $t > 0$ and for $x \in \Omega$.*

We also get from Proposition 1 a quantitative expression of the continuous dependence of solution H of $P(\mu, Q_0)$ with respect to the data Q_0 and H_0.

Corollary 2 *Let H_1 and H_2 be the solutions of $P(\mu, Q_0)$ corresponding to $\mu \geq 0$, $Q_{1,0}$, $Q_{2,0}$ and $H_{1,0}$, $H_{2,0}$ respectively. Then for any $t > 0$*

$$\int_\Omega |H_1(t, x) - H_2(t, x)|\, dx \leq \int_\Omega |H_{1,0}(x) - H_{2,0}(x)|\, dx + \int_0^t |(Q_{1,0}(\tau) - Q_{2,0}(\tau))|\, d\tau.$$

Proof of Corollary 2. It is enough to observe that, for any general function $a(x)$ defined on Ω we have that $|a(x)| = a_+(x) + a_-(x)$ and that, if for fixed $t > 0$ we define $a(x) = H_1(t, x) - H_2(t, x)$ then $a_-(x) = -\min(0, a(x)) = (H_2(t, x) - H_1(t, x))_+$. Since, the order of H_1 and H_2, taken in Proposition 1, is arbitrary, by reversing the roles of H_1 and H_2, we get that

$$\int_\Omega (H_1(t,x) - H_2(t,x))_- dx \leq \int_\Omega (H_{1,0}(x) - H_{2,0}(x))_- dx + \int_0^t (Q_{1,0}(\tau) - Q_{2,0}(\tau))_- d\tau, \quad (13)$$

which concludes the proof.

Proof of the Proposition 1. The main idea is to multiply the difference of the two equations by a regular approximation $p_n(r)$, $n \in \mathbb{N}$, of the Heaviside type function

$$sign_{+,0}(r) = 0 \text{ if } r \leq 0 \text{ and } sign_{+,0}(r) = 1 \text{ if } r > 0,$$

taking as $r = (H_1^2(t, x) - H_2^2(t, x))$. For instance, we can take p_n

$$p_n(r) = \begin{cases} 0 & \text{if } r \leq -\frac{1}{n}, \\ nr & \text{if } r \in [-\frac{1}{n}, \frac{1}{n}], \\ 1 & \text{if } r > \frac{1}{n}. \end{cases} \quad (14)$$

Then,

$$\int_\Omega \left(\frac{\partial H_1(t,x)}{\partial t} - \frac{\partial H_2(t,x)}{\partial t}\right) p_n(H_1^2(t,x) - H_2^2(t,x)) dx = K \int_\Omega \frac{\partial}{\partial x}\left(\left(\frac{\partial}{\partial x} H_1^2(t,x) - \frac{\partial}{\partial x} H_2^2(t,x)\right) p_n(H_1^2(t,x) - H_2^2(t,x))\right) dx + \mu \int_\Omega \frac{\partial}{\partial x}(H_1^\lambda(t,x) - H_2^\lambda(t,x)) p_n(H_1^2(t,x) - H_2^2(t,x)) dx.$$

By the definition of weak solution (i.e., by integrating by parts) we get

$$\int_\Omega \left(\frac{\partial H_1(t,x)}{\partial t} - \frac{\partial H_2(t,x)}{\partial t}\right) p_n(H_1^2(t,x) - H_2^2(t,x)) dx + K \int_\Omega \left(\frac{\partial}{\partial x} H_1^2(t,x) - \frac{\partial}{\partial x} H_2^2(t,x)\right)^2 p_n'(H_1^2(t,x) - H_2^2(t,x)) dx = \mu \int_\Omega \frac{\partial}{\partial x}(H_1^\lambda(t,x) - H_2^\lambda(t,x)) p_n(H_1^2(t,x) - H_2^2(t,x)) dx + K\left(\frac{\partial}{\partial x} H_1^2(t,0) - \frac{\partial}{\partial x} H_2^2(t,0)\right) p_n(H_1^2(t,0) - H_2^2(t,0)),$$

where we used the facts that support of $H_1^2(t,.) - H_2^2(t,.)$ is a compact set for any $t \geq 0$. Then, since $0 \leq p_n(r) \leq 1$ for any r, and passing to the limit, as $n \to +\infty$ we have that

$$sign_{+,0}(H_1^2(t,x) - H_2^2(t,x)) = sign_{+,0}(H_1(t,x) - H_2(t,x)) = sign_{+,0}(H_1^\lambda(t,x) - H_2^\lambda(t,x)).$$

Finally, it is enough to remember that

$$\frac{\partial H(t,x)}{\partial t} sign_{+,0}(H(t,x)) = \frac{\partial [H(t,x)]_+}{\partial t} \text{ and } \frac{\partial H(t,x)}{\partial x} sign_{+,0}(H(t,x)) = \frac{\partial [H(t,x)]_+}{\partial x}$$

for any general function $H(t,x)$ and so the result follows by interating in t and using that support of $H_1^\lambda(t,.) - H_2^\lambda(t,.)$ is a compact set for any $t \geq 0$ and that $[H_1^\lambda(t,0) - H_2^\lambda(t,0)]_+ \geq 0$.

3 Limited volcanoes base for $\mu > 0$.

Concerning the theory of existence and uniqueness of weak solutions we send the reader to the works [1], [10], [6], [9], [8], [4] and their references.

Theorem 2 *Assume $H_0(x)$ bounded and with compact support,*

$$0 < \lambda < 2,$$

and let

$$0 \leq Q_0(t) \leq Q_{0,\infty}, \text{ for any } t > 0 \tag{15}$$

for a suitable $Q_{0,\infty}$. Then the support $H(t,\cdot) = [-\xi(t), 0) \cup (0, \xi(t)]$, for any $t > 0$,

$$|\xi(t)| \leq \xi_\infty \text{ for any } t \geq 0,$$

for some finite $\xi_\infty < \infty$ depending on $\lambda, K, \mu, Q_{0,\infty}$ and $H_0(x)$.

Proof. Thanks to Corollary 1 it is enough to construct a supersolution $H_2(t,x)$ with a bounded support for any $t \geq 0$. In fact, we can construct such a function as $H_2(t,x) = U(x)$ solution of the ordinary differential equation

$$\begin{cases} K(U^2)_x + C_1 U^\lambda = 0 \quad x \in (0, +\infty), \\ U(0) = C_2. \end{cases}$$

Using that $\lambda < 2$ the support of U is compact and since $H(t,x)$ is bounded we can choose $C_1, C_2 > 0$ suitably as to have

$$Q_{1,0}(t) \leq C_1 C_2^\lambda \text{ for any } t > 0 \text{ and } H_{1,0}(x) \leq U(x) \text{ for } x \in \Omega,$$

and the proof is complete.

Remark. Other supersolutions leading to other qualitative properties of the free boundary can be found in the works [10], [6], [7], [8], [5] and [3].

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